

Connection between chiral symmetry restoration and deconfinement

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We propose a simple explanation for the connection between chiral symmetry restoration and deconfinement in QCD at high temperature. In the Higgs description of the QCD vacuum both spontaneous chiral symmetry breaking and effective gluon masses are generated by the condensate of a color octet quark-antiquark pair. The transition to the high-temperature state proceeds by the melting of this condensate. Quarks and gluons become (approximately) massless at the same critical temperature. For instanton-dominated effective multi-quark interactions and three light quarks with equal mass we find a first order phase transition at a critical temperature around 170 MeV.

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I. OCTET MELTING

Lattice simulations for QCD at high temperature [1] find empirically that chiral symmetry restoration and deconfinement occur at the same critical temperature T_c [2]. Sufficiently below T_c the equation of state is rather well approximated by a gas of mesons. Gluons and quarks or baryons play no important role. Above T_c the dominant thermodynamic degrees of freedom are gluons and quarks. The change in the relevant number of degrees of freedom occurs rather rapidly and may be associated with a thermodynamic phase transition [3].

The transition to the high-temperature state of QCD is often called the “deconfinement transition” since the gluon modes with momenta $p^2 \approx (\pi T)^2$ characteristic for a thermal state behave at high temperature close to a gas of free particles. (For experimentally accessible temperatures above the phase transition substantial corrections to the equation of state of a relativistic gas remain present.) This also holds for the fermionic degrees of freedom which appear essentially as free quarks. Having a closer look, the picture of weakly interacting gluons is actually somewhat oversimplified. In fact, the long-distance behavior of the time-averaged correlation functions in high-temperature QCD corresponds to a “confining” three-dimensional effective theory [4] with a strong effective coupling and a temperature-dependent “confinement scale” [5,6]. Furthermore, the transition between confinement and weakly interacting “free” quarks is less dramatic than it may seem at first sight. In particular, it need not be associated with a true phase transition. In the presence of light quarks, the temperature-induced changes in the long-distance behavior of the heavy quark potential are only quantitative [7]. String breaking occurs for arbitrary temperature.

Nevertheless, one expects a temperature range where the properties of strong interactions change rapidly, reflecting “deconfinement.” We associate “deconfinement” here with the transition to a thermal equilibrium state for which (1) the bosonic contribution to the free energy can be approximated by the one of a free gas of eight massless spin-one bosons and, correspondingly, (2) possible masslike terms in the ef-

fective “gluon propagator” are of the order of the temperature or smaller. We do not deal in this paper with temperature-dependent changes in the heavy quark potential and temperature effects in the dynamics of string breaking.

On the other hand, the chiral properties of QCD are also expected to undergo a rapid change at some temperature. The vacuum is characterized by a chiral condensate of quark-antiquark pairs. This order is destroyed at high temperature and chiral symmetry is restored. For vanishing quark masses this implies a true phase transition. In this case chiral symmetry restoration at high temperature is signaled by a vanishing order parameter. At high temperature the massless quarks contribute to the free energy as a relativistic gas with $3N_f$ fermions whereas below a critical temperature the fermionic degrees of freedom appear as massive baryons without much relevance for thermodynamics. This qualitative change in the fermionic contribution remains true also in presence of small nonvanishing quark masses. We associate the chiral transition at high T with a rapid decrease of jump in the chiral order parameter and the appearance of a relativistic quark gas contribution to the free energy.

The qualitative feature that at high-temperature deconfinement and chiral symmetry restoration come in pairs is not much of a mystery: modes with momenta $\sim T$ dominate the thermal state and the interactions between these modes become comparatively weak. Symmetry restoration at high T is a common phenomenon. It is a longstanding puzzle, however, why deconfinement and chiral symmetry restoration occur quantitatively at the same temperature. Analytical approaches that have been investigated so far either concentrate on the quark degrees of freedom—for example, in the context of Nambu–Jona-Lasinio models [8]. This reflects important aspects of the chiral transition but fails to describe the effects of confinement and the thermodynamic equation of state of QCD. Other models deal with pure QCD without including the important aspects connected with the presence of light quarks. So far we are not aware of any successful quantitative analytical description accounting simultaneously for the deconfinement and chiral symmetry restoration aspects of the QCD phase transition. In view of the vast experimental efforts [9] and the important progress in numerical simulations [10,7] even an oversimplified analytical

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description of the most salient characteristics of this transition would be very welcome.

One of the main difficulties for any analytical description are the different relevant degrees of freedom above and below the critical temperature. Whereas for high T a gas of quarks and gluons becomes a reasonable approximation, the low temperature physics is described by an interacting pion gas with far fewer degrees of freedom. The fermionic degrees of freedom at low T are baryons which are Boltzmann suppressed. Any quantitative analytical description must be able to describe both gluons and pions as well as quarks and baryons simultaneously. This is mandatory at least in the vicinity of the critical temperature which is precisely characterized by the transition from one set of dominant degrees of freedom to another. The recently proposed Higgs picture of the QCD vacuum [11,12] offers such a possibility since all relevant degrees of freedom are described at once. A reasonable picture of the QCD-phase transition may also serve as a test for these ideas.

In this paper we propose a simple explanation of the simultaneous occurrence of the deconfinement and chiral transition and present first quantitative estimates. Our explanation is based on the proposed new understanding of the QCD vacuum by “spontaneous breaking of color” or gluon-meson duality [11]. In this picture of the vacuum the condensate $\langle\chi\rangle$ of a color octet quark-antiquark pair leads to spontaneous breaking of the color symmetry. (As for the electroweak standard model there exists an equivalent gauge invariant formulation.) The gluons acquire a mass $\sim\langle\chi\rangle$ by the Higgs mechanism and become integer charged. Gluon-meson duality associates the massive gluons with the physical vector mesons ρ , K^* , and ω . This is possible since they carry the appropriate integer electric charges as well as isospin and strangeness. Also the quarks carry integer charges after spontaneous color symmetry breaking and can be associated with the baryons $(p, n, \Lambda, \Sigma, \Xi)$ plus a heavy singlet. They become massive due to the spontaneous breaking of the chiral flavor symmetry by $\langle\chi\rangle$ and a similar singlet $\bar{q}q$ condensate. This association between quark fields and baryons is called quark-baryon duality. It is the analogue of color-flavor locking [13] for a high baryon density.

It has been argued on phenomenological grounds [11] that the dominant contribution to spontaneous chiral symmetry breaking arises from the octet condensate $\langle\chi\rangle$. The same condensate therefore explains both confinement and spontaneous chiral symmetry breaking. In this description the solution to the puzzle why deconfinement and chiral symmetry restoration are connected becomes obvious. At some critical temperature T_c the value of the octet condensate $\langle\chi\rangle$ drops rapidly. At this temperature the mass of the gluons therefore drastically decreases and the deconfinement transition happens. In the limit of three massless quarks the expectation value $\langle\chi\rangle$ exactly vanishes at T_c , reflecting chiral symmetry restoration. Typically, the color-octet and -singlet $\bar{q}q$ -expectation values influence each other. In particular, an octet condensate always induces a singlet condensate. It seems plausible that both the octet and singlet condensates vanish simultaneously for $T>T_c$. Then chiral symmetry gets

completely restored for $T>T_c$, leading to a true phase transition characterized by the order parameter of chiral symmetry breaking. The critical temperature for the phase transition is the “melting temperature” for the octet condensate. The “deconfinement temperature” and “chiral symmetry restoration temperature” are therefore identical. As a consequence of the change in the order parameter the quantum numbers of the excitations in the “hadronic phase” for $T<T_c$ differ from the ones in the “quark-gluon phase” for $T>T_c$.

Quark mass effects modify the details of the transition. For a more quantitative description of “octet melting” we omit in this paper the difference between the current quark masses and consider three light quark flavors with equal mass. For our quantitative analysis we will need information about the condensates in the vacuum which are related to the form of the effective quark interactions. We investigate a rather wide class of effective multi-quark interactions induced by instanton effects. In the chiral limit of three massless quarks we find a first-order transition with critical temperature $T_c \approx 130-160$ MeV. In this limit the mass of the eight pseudoscalar Goldstone bosons (π, K, η) vanishes at low temperature $m_{PS}=0$. For realistic current quark masses corresponding to a nonzero average pseudoscalar mass $m_{PS}^2 \approx (2M_K^2 + M_\pi^2)/3$, we again get a first-order transition with a somewhat larger $T_c \approx 170-180$ MeV.

In this note we do not deal explicitly with the color-singlet quark-antiquark pair which plays a subdominant role. The relevant thermodynamic potential is then given by the temperature-dependent effective potential for the octet condensate

$$U(\chi, T) = U_0(\chi) + \Delta U(\chi, T). \quad (1)$$

Here U_0 encodes in a bosonic language for scalar $\bar{q}q$ composites the information about the multi-quark interactions in the vacuum, whereas ΔU accounts for the thermal fluctuations. We use a scalar field $\chi_{ij,ab} \sim \bar{\psi}_{jb}\psi_{ai} - \frac{1}{3}\bar{\psi}_{kb}\psi_{ak}\delta_{ij}$ for the octet quark-antiquark pair with $i, j = 1 \dots 3$ color indices and $a, b = 1 \dots N_f$ flavor indices for massless (or light) quarks. For $N_f=3$ it is sufficient to evaluate the effective potential for the direction of the condensate [11]

$$\langle\chi_{ij,ab}\rangle = \frac{1}{\sqrt{6}}\chi\left(\delta_{ia}\delta_{jb} - \frac{1}{3}\delta_{ij}\delta_{ab}\right). \quad (2)$$

In the limit of three equal quark masses this condensate preserves a vectorlike $SU(3)$ symmetry which contains the generators of isospin and strangeness.

We emphasize that in our picture the thermodynamic quantities depend on the number of light flavors in an important way. For example, the vacuum condensates for two-flavor QCD ($N_f=2$) are discussed in Ref. [12] and differ substantially from the three-flavor case. Needless to say that in absence of light quarks the chiral aspects of the phase transition are completely different. We restrict our discussion here to the realistic case $N_f=3$. Quark mass effects are taken into account except for the $SU(3)$ violation due to the mass differences. The thermodynamic quantities of interest can

then be extracted from the behavior of the minima of $U(\chi, T)$. For $T=0$ the minimum occurs for a nonvanishing octet condensate $\chi_0 \neq 0$. As the temperature increases, the thermal fluctuations induce a new local minimum at the origin $\chi=0$. At the critical temperature T_c the two minima are degenerate. The discontinuity characteristic for a first-order phase transition corresponds to the jump to the absolute minimum at $\chi=0$ for $T>T_c$.

The remainder of this paper is devoted to a more detailed estimate of $U(\chi, T)$. We will employ a mean-field-type calculation which takes into account, nevertheless, the most important effects of the strong interactions. We will devote some care to the proper implementation of the mean field calculation despite the fact that some other approximations in our approach (such as the effective vacuum potential) remain rather crude. The reason is that we want to describe correctly the interacting pion gas at low temperature within a linear formulation. The linear formulation is necessary for a description of symmetry restoration at T_c , since for $T>T_c$ the Goldstone bosons are combined with the σ scalar into linear multiplets of the flavor group. On the other hand, the nonlinear Goldstone excitations dominate the free energy at low T and chiral perturbation theory gives a valid description. We will see that a linear mean field description easily leads to incorrect results for the nonlinear excitations unless done with sufficient care. Furthermore, we do not want to obscure the possibility of a second order phase transition by using a method which is too rough. Again, unless the bosonic degrees of freedom are treated with care, a mean field computation can easily produce a spurious first order jump even in a situation of a second order transition or a crossover. This is well known from the study of scalar field theories or the electroweak phase transition.

Our paper is organized as follows. Section II addresses the form of the vacuum potential $U_0(\chi)$, which was estimated previously [14] by an instanton calculation. In Sec. III we discuss the dependence of the masses of the most relevant excitations on the octet condensate χ . This is the basis for our mean-field-type calculation of the temperature effects $\Delta U(\chi, T)$ in Sec. IV. A first summary of quantitative results for the chiral phase transition is given in Sec. V in the chiral limit of massless quarks. For the instanton-induced potential $U_0(\chi)$ proposed in Sec. II we find a critical temperature $T_c = 154$ MeV, consistent with lattice estimates [7]. Section VI is devoted to the high-temperature phase and the equation of state. Sufficiently above T_c one observes a weakly interacting gas of gluons and quarks. In Sec. VII we present analytical estimates for the chiral phase transition. In particular, for three massless quarks the critical temperature is related to the mass of the η' meson and its decay constant f_θ by

$$T_c^4 = 0.013 M_{\eta'}^2 f_\theta^2 R^{-1/4} \quad (3)$$

with $R < 1$ a constant of order 1.

We turn to the value of the chiral symmetry-breaking order parameter in the low temperature phase in Sec. VIII. This needs a meaningful treatment of the Goldstone boson fluctuations which we propose in Sec. IX. In Sec. X we verify that our prescriptions are consistent with chiral perturbation

theory for low enough T . Section XI discusses the effects of nonvanishing current quark masses. We consider equal current quark masses which correspond to an average mass of the pseudoscalar Goldstone bosons $M_{PS} = 390$ MeV. Up to $SU(3)$ -violating quark mass differences this corresponds to a realistic situation. We find $T_c = 170$ MeV, again consistent with lattice simulations [7]. In the low temperature phase the Higgs contribution to the mass of the vector mesons decreases substantially as the temperature increases. In the vicinity of T_c we find a Higgs contribution to the average vector meson mass ≈ 300 MeV and to the average mass for the light baryons ≈ 600 MeV. Those have to be supplemented by thermal mass contributions. A decrease of the vector meson mass [15] may be relevant for the dilepton spectrum observed in heavy ion collisions. Finally, we investigate in Sec. XII a possible lattice test of our scenario by establishing the existence of stable Z_3 vortices in the low-temperature phase. Section XIII summarizes our conclusions.

II. VACUUM-EFFECTIVE POTENTIAL

The effect of the thermal fluctuations has to be compared with the vacuum-effective potential $U_0(\chi)$ for the octet. We will not need here many details of the precise shape of U_0 . The most important parameter for the determination of the critical temperature T_c for the phase transition will turn out to be the difference $U_0(0) - U_0(\chi_0)$, with χ_0 the vacuum expectation value of the octet. To be specific, we concentrate here on an instanton-induced potential which is motivated by the observation that instanton effects are a plausible candidate for the dynamics that lead to a nonvanishing octet condensate [14]. Beyond the octet $\bar{q}q$ -condensate realistic QCD exhibits also a nonvanishing color-singlet $\bar{q}q$ -condensate σ . In order to concentrate our discussion on the essential aspects, we assume here that σ is “integrated out” by solving its effective field equation¹ as a functional of χ . We take here $\sigma \sim \chi$ (in the relevant range) such that all $\bar{q}q$ condensates are accounted for by χ .

For $N_f = 3$ and small values of $\bar{q}q$ the instanton effects lead to a cubic term $U_{an} \sim \chi^3 \sim (\bar{q}q)^3$. For large values of χ the nonvanishing gluon mass $\sim \chi$ leads to an effective infrared cutoff for the instanton interactions, resulting in $U_{an} \sim \chi^{-11}$ [14]. A first quantitative estimate of the instanton-induced effective potential can be found in Ref. [14] and we exploit its consequences in the present work. However, in view of the uncertainties of this estimate we will be satisfied here with a simplified version which respects the asymptotic properties for small and large χ .

¹The effective kinetic term and interactions of the χ field include therefore contributions from the color singlet field σ as well. In a two-field language a nonzero σ is necessarily generated for $\chi \neq 0$ by terms $\sim \sigma \chi^2$ present in the contribution from the chiral anomaly for $N_f = 3$.

$$(A): U_0(\chi) = \lambda \left\{ -\frac{14}{11} R_{an} \chi_0 \chi^3 \left[1 + \frac{3}{11} \left(\frac{\chi}{\chi_0} \right)^{14} \right]^{-1} + (1 - R_{an})(\chi^4 - 2\chi_0^2 \chi^2) + \chi_0^4 \right\}. \quad (4)$$

The instanton contribution is the one $\sim R_{an}$ and we have added another contribution $\sim (1 - R_{an})$ which arises from nonanomalous interactions not related to instantons, such as, for example, multi-quark interactions mediated by gluon exchange. Presumably the vacuum potential is dominated by instanton effects [14] such that R_{an} is expected to be close to 1. In our conventions χ is real and we have parametrized the potential such that the absolute minimum of U_0 occurs at χ_0 , with $U_0(\chi=0) = \lambda \chi_0^4, \lambda > 0$. The additive constant is fixed such that the pressure vanishes in the vacuum. In the absence of quark masses this implies $U_0(\chi_0) = 0$.

In order to investigate how sensitively our results depend on the precise shape of the potential, we consider for $N_f = 3$ also a potential which only incorporates the breaking of the discrete symmetry $\chi \rightarrow -\chi$ by the anomaly, without having the correct asymptotic behavior

$$(B): U_0(\chi) = \lambda \{ R_{an}(3\chi^4 - 4\chi_0 \chi^3) + (1 - R_{an})(\chi^4 - 2\chi_0^2 \chi^2) + \chi_0^4 \}. \quad (5)$$

Finally, we compare our results also to a quartic potential

$$(C): U_0(\chi) = \lambda(\chi^2 - \chi_0^2)^2. \quad (6)$$

The quartic potential (C) is characteristic for two-flavor QCD ($N_f = 2$) (or for three flavors with small effects of the chiral anomaly). We emphasize that the investigation of the potentials (B) and (C) serves mainly the purpose of an estimate of uncertainties or errors from the unknown details of the shape of U_0 whereas we consider (A) as a more realistic potential.

As we have already mentioned, the dominant parameter for the characteristics of the chiral phase transition is $U_0(0)$. This measures in our convention the potential difference between the chiral symmetric state at $\chi=0$ and the vacuum at $\chi=\chi_0$. It gets a contribution from the chiral anomaly $U_{an}(0)$ and we have in our parametrization

$$U_0(0) = R_{an}^{-1} U_{an}(0). \quad (7)$$

Here we denote by $U_{an}(\chi)$ the contribution of the chiral anomaly, i.e., the part $\sim R_{an}$ in Eqs. (4) or (5). The size of the chiral anomaly in the vacuum $U_{an}(\chi_0) - U_{an}(0)$ is given for our simple potential (4) by $-U_{an}(0)$. It is directly related [11,16] to the mass of the η' meson $M_{\eta'} = 960$ MeV and its decay constant $f_\theta \approx 150$ MeV

$$U_{an}(0) = \frac{f_\theta^2 M_{\eta'}^2}{2N_f^2} = \frac{1}{N_f^2} \times 10^{-2} \text{ GeV}^4. \quad (8)$$

The parameter R_{an} is essentially fixed for instanton-dominated multi-quark interactions ($R_{an} \approx 1$). This sets the scale for U_0 in terms of the mass and decay constant of the

η' meson. This is the basis for an interesting connection between the critical temperature T_c on one side and $M_{\eta'}$ and f_θ on the other side which will become apparent later.

The vacuum octet condensate χ_0 is related to the effective gauge coupling g and the vector meson mass $\bar{\mu}_\rho$ by $\bar{\mu}_\rho = g(\bar{\mu}_\rho) \chi_0$. The coupling $g(\bar{\mu}_\rho)$ can be estimated [11] from the ρ -decay width into two pions or charged leptons and we choose $g(770 \text{ MeV}) = 6$. For $\bar{\mu}_\rho = 770$ MeV this yields $\chi_0 \approx 130$ MeV. [In view of the uncertainties of this estimate one could also treat $g(\bar{\mu}_\rho)$ as an unknown parameter within a certain range around the estimate above.] The coupling λ is now determined by

$$\lambda = \frac{R_{an}^{-1} f_\theta^2 M_{\eta'}^2}{2N_f^2 \chi_0^4} = \frac{g^4(\bar{\mu}_\rho)}{2N_f^2 R_{an}} \frac{f_\theta^2 M_{\eta'}^2}{\bar{\mu}_\rho^4}. \quad (9)$$

Another characteristic quantity for the vacuum potential is the mass of the σ resonance given by²

$$M_\sigma^2 = \frac{3}{8} \partial^2 U_0 / \partial \chi^2|_{\chi_0}. \quad (10)$$

Phenomenology indicates $M_\sigma \approx 500$ MeV for realistic values of the quark masses.

III. EFFECTIVE MASSES

We will evaluate the temperature-dependent part of the effective potential $\Delta U(\chi, T)$ in a generalized mean-field-type approximation. In a strong interaction environment reasonable results can only be obtained if the dependence of the effective particle masses on the mean field and the momentum scale is dealt with some care. In particular, the strong running of the couplings should not be neglected. Furthermore, we want to be able, in principle, to distinguish between a first or second-order phase transition. In this respect it is crucial that the Goldstone bosons are treated correctly, since otherwise a second-order transition may be obscured by an insufficient approximation. This is not trivial in a mean field treatment of a linear model and we devote some attention to this issue in Secs. IX–XI.

More precisely, we include the effects of the fluctuations of the gluons, quarks, and pseudoscalar Goldstone bosons. Since in the low temperature phase gluons are associated to vector mesons and quarks to baryons [11], these fluctuations include all light particles. Our approximation is formally equivalent to a gas of noninteracting massive particles. However, the effects of the interactions appear indirectly through the dependence of the particle masses on the mean field and the momentum scale. In particular, the gluons and quarks have χ -dependent masses

$$\mu_\rho^2(\mu) = g^2(\mu) \chi^2, \quad M_q^2(\mu) = h_\chi^2(\mu) \chi^2, \quad (11)$$

²This holds if the contribution of the color singlet to the kinetic term of χ is small.

where we take into account the running of the gauge and Yukawa coupling. The running of the couplings and the choice of the renormalization scale μ are specified in Appendix A. From the average baryon mass in vacuum $\bar{M}_q = h_\chi(\bar{\mu}_\rho)\chi_0 = 1150$ MeV we extract³ $h_\chi(\bar{\mu}_\rho) = 9$. Beyond the running couplings further effects of the interactions of the gluons are omitted. We neglect here, in particular, the thermal mass corrections which contribute $\sim gT$ to a chirally invariant quark mass and the gluon mass. (For simplicity, we also have neglected the mass splitting between the baryon octet and the singlet which are described by the nine quarks.)

For the Goldstone bosons the squared mass is given for all T by the derivative of the potential

$$M_G^2 = c_G U' = c_G \frac{\partial U}{\partial(\chi^2)} = \frac{36}{7N_f} \frac{\partial U}{\partial(\chi^2)}. \quad (12)$$

Also the effective meson decay constant depends on χ

$$f^2 = c_F \chi^2 = \frac{7}{9} \chi^2. \quad (13)$$

As chiral symmetry restoration is approached for $\chi \rightarrow 0$ the effective decay constant decreases. The same holds for the chiral condensate $\langle \bar{\psi}\psi \rangle \sim \chi$. We note that the general relation

$$f^2 M_G^2 = \frac{2\chi}{N_f} \frac{\partial U}{\partial \chi} \quad (14)$$

implies

$$c_G c_F = \frac{4}{N_f} \quad (15)$$

independently of the individual constants⁴ c_G and c_F .

IV. THERMAL FLUCTUATIONS

The various contributions of the thermal fluctuations are now evaluated as

$$\Delta U = \Delta_g U + \Delta_q U + \Delta_s U \quad (16)$$

with

$$\begin{aligned} \Delta_g U &= 24 J_B (g^2 \chi^2), \quad \Delta_q U = -12 N_f J_F (h_\chi^2 \chi^2), \\ \Delta_s U &= (N_f^2 - 1) [J_G (M_G^2) + \Delta J_G]. \end{aligned} \quad (17)$$

Here we have defined the integrals

³Note that actually only $M_q(\chi=0)=0$ and $M_q(\chi=\chi_0)=\bar{M}_q$ are known. The χ dependence of $M_q(\chi)$ could be more complicated than the ansatz used in Eq. (11).

⁴The values used in Eqs. (12) and (13) neglect the contribution of the singlet $\bar{q}q$ condensate.

$$J(M^2, T) = \frac{1}{2} \left(\int_q^{(T)} \ln(q_T^2 + M^2) - \int_q^{(0)} \ln(q^2 + M^2) \right) \quad (18)$$

with $\int_q^{(T)} = T \sum_n \int [d^3 \vec{q} / (2\pi)^3]$, $\int_q^{(0)} = \int [d^4 q / (2\pi)^4]$, $q_T^2 = (2\pi n T)^2 + \vec{q}^2$, $q^2 = q_0^2 + \vec{q}^2$. For the bosonic fluctuations J_B and J_G the sum is over integer Matsubara frequencies n . In contrast, for J_F the Matsubara frequencies are half integer and therefore $q_T^2 = (2n+1)\pi T$, $n \in \mathbb{Z}$. The integrals J_B and J_F reflect the difference between the one loop expressions between nonzero and zero temperature for a given “background field” χ . They are well known in thermal field theory [17]

$$\begin{aligned} J_B(M^2, T) &= T \int_0^\infty \frac{dq q^2}{2\pi^2} \ln[1 - \exp(-\sqrt{q^2 + M^2}/T)], \\ J_F(M^2, T) &= T \int_0^\infty \frac{dq q^2}{2\pi^2} \ln[1 + \exp(-\sqrt{q^2 + M^2}/T)], \end{aligned} \quad (19)$$

and can be expressed in terms of dimensionless integrals with $m^2 = M^2/T^2$

$$J_{B(F)} = \frac{T^4}{2\pi^2} \int_0^\infty d\tilde{q} \tilde{q}^2 \ln[1 \mp \exp(-\sqrt{\tilde{q}^2 + m^2})]. \quad (20)$$

We observe the Boltzmann suppression for large m , i.e.,

$$\lim_{m \rightarrow \infty} J_{B(F)} = (\mp) \frac{T^4}{2\sqrt{2}} \left(\frac{m}{\pi} \right)^{3/2} e^{-m}. \quad (21)$$

The fluctuation integral J_G for the Goldstone bosons has to be handled with care since M_G^2 may be negative for a certain range of χ . The Goldstone boson mass M_G^2 vanishes at the temperature-dependent minimum of the potential $\chi_0(T)$. [In our notation $\chi_0(0) \equiv \chi_0$ corresponds to the parameter appearing in the zero-temperature effective potential (5).] The physics of the “nonconvex region” $\chi < \chi_0(T)$, where our approximation leads to $\partial U / \partial(\chi^2) < 0$, can be properly discussed [18] only in the context of a coarse-grained effective action as the effective average action [19,20]. We give here some “ad hoc” prescriptions which catch the relevant physics for most (not all) situations. For negative U' the fluctuations with momenta $q^2 < -U'$ should be replaced by spin waves, kinks or similar “tunneling” configurations which fluctuate between different minima of the potential [18]. Here we simply omit the fluctuations with $q^2 < -U'$. As a consequence, J_G equals J_B for $U' \geq 0$, whereas the lower integration boundary for \tilde{q} in Eq. (20) is replaced by $|m|$ if m^2 is negative. With

$$J_G(M^2, T) = \begin{cases} J_B(M^2, T) & \text{for } M^2 \geq 0, \\ \frac{T^4}{2\pi^2} \int_0^\infty dy y \sqrt{y^2 + |m^2|} \ln(1 - e^{-y}) & \text{for } M^2 < 0 \end{cases} \quad (22)$$

we observe that J_G is continuous in M^2 . For $M^2 < 0$ the contribution of the Goldstone fluctuations is enhanced as compared to a gas of massless free particles. Finally, the correction ΔJ_G in Eq. (17) is related to the appropriate treatment of the running couplings and specified in Appendix A.

Our treatment of the composite scalar fluctuations leads to some shortcomings. The perhaps most apparent one concerns the behavior of ΔU for very high T . In this region the gluons have only two helicities and the factor 24 multiplying J_B should be replaced by 16. In fact, a more correct formulation would treat the $2(N_c^2 - 1)$ transversal gluon degrees of freedom separately from the longitudinal degrees of freedom. The latter ones are associated to the scalar sector by the Higgs mechanism. In contrast to the fundamental scalar in the electroweak sector of the standard model the octet χ is a composite and the scalars should not be counted as independent degrees of freedom at high T . In a more complete treatment this is realized by large T -dependent mass terms for all scalars which suppress their contribution effectively for large T . The neglect of this effect in our rough treatment leads to an overestimate of the bosonic contribution⁵ at large T by a factor 3/2.

On the other hand, in the low temperature region we have neglected bosonic contributions from the σ resonance or the scalar octet (a^\pm , etc.) which have a similar mass as the vector mesons. This results in an underestimate of the bosonic contribution which is particularly serious near a second-order or weak first-order phase transition. At a second-order phase transition all scalars contained in an appropriate linear representation become massless. In the chiral limit the minimal set are 144 real scalars for $N_f=3$ and 32 real scalars for $N_f=2$. In contrast, our treatment accounts only for $(N_c^2 - 1) + (N_f^2 - 1)$ massless scalars at the phase transition.

Another shortcoming is the neglect of temperature effects in the instanton-induced interactions. They become important for $\pi T \approx \mu_\rho$ and can be neglected for low T . It is conceivable that these effects influence substantially the details of the phase transition. Despite all that we remind that the gross features of the equation of state and the value of the critical temperature are relatively robust quantities. We will see below that an error of 20% in the number of effective degrees of freedom in the quark-gluon phase at T_c influences the value of T_c only by 5%. A similar statement holds for a 20% error in the estimate of $U_0(0) - U_0(\chi_0)$. For a first rough picture of the implications of the color octet condensate on the chiral phase transition our approximation of the thermal fluctuations (16) seems therefore appropriate.

⁵The suppression of the Goldstone boson contribution J_G at large T is included properly in our approximation.

TABLE I. First-order phase transition in the chiral limit ($m_{PS} = 0$).

U_0	vacuum			critical temperature			
	$\bar{\mu}_\rho$	f	M_σ	T_c	$\bar{\mu}_\rho^{(SSB)}(T_c)$	$M_q^{(SSB)}(T_c)$	$\tau(T_c)$
A	700	68	1570	154	290	580	0.37
B	770	113	580	134	440	700	0.66
C	770	113	480	131	450	720	0.71

V. HIGH-TEMPERATURE PHASE TRANSITION FOR VANISHING QUARK MASSES

For a given form of the vacuum potential U_0 the temperature dependence of the effective potential and the temperature-dependent masses of the pseudoparticles can now be computed. We concentrate here on a potential that is dominated by instanton effects reflecting the chiral anomaly and take $R_{an}=0.9$ in Eq. (4). In Table I we present for the chiral limit (vanishing current-quark masses) the values of the vector-meson mass $\bar{\mu}_\rho$ (input), the decay constant f and the σ mass M_σ for the vacuum. The results for the instanton-induced potential are denoted by (A). For our parameters realistic values of $\bar{\mu}_\rho$, f , and M_σ are obtained for a realistic average current quark mass (see Sec. XI, Table II). They differ from the chiral limit shown in Table I. In order to investigate the influence of the uncertainties in the vacuum potential on the thermal properties, we also compare with corresponding results for the potentials (B), (C) in Eqs. (5), (6). [For the potentials (B),(C) we have not optimized the input parameters $\bar{\mu}_\rho, g(\bar{\mu}_\rho), h_\chi(\bar{\mu}_\rho)$ with respect to the observed particle masses for nonzero current quark masses.]

As the temperature increases, we find a first-order phase transition. The critical temperature T_c is indicated in Table I, as well as the Higgs contribution to the mass of the vector mesons and baryons slightly below the critical temperature [$\bar{\mu}_\rho^{(SSB)}(T_c)$ and $M_q^{(SSB)}(T_c)$]. We observe that the decrease of the effective masses $\bar{\mu}_\rho(T)$ and $M_q(T)$ for increasing temperature is largely determined by the temperature dependence of the effective gauge and Yukawa couplings and only to a smaller extent by the decrease of the expectation value $\chi_0(T)$. We observe that the true screening masses get temperature corrections which increase $\sim T$ and become important near T_c . Also the maximum of the vector meson spectral function, which is relevant for dilepton production, differs from the screening mass. As T increases beyond T_c , the absolute minimum of $U(\chi, T)$ jumps to $\chi=0$ and the gluons

TABLE II. First-order phase transition for realistic average quark masses ($m_{PS}=390$ MeV).

U_0	vacuum			critical temperature			
	$\bar{\mu}_\rho$	f	M_σ	T_c	$\bar{\mu}_\rho^{(SSB)}(T_c)$	$M_q^{(SSB)}(T_c)$	$\tau(T_c)$
A	770	116	460	170	290	600	0.53
B	800	132	770	180	470	800	0.77
C	810	142	660	183	490	840	0.77

and quarks (or vector-mesons and baryons) become massless in our approximation. The equation of state approaches rapidly the one for a free gas of gluons and quarks (see Sec. VI for details).

Since the potentials (B),(C) are qualitatively quite different from (A), we believe that the range $T_c \approx 130\text{--}160$ MeV roughly covers the uncertainty from the choice of the vacuum potential U_0 . We note that the critical temperature for the instanton-induced potential (A) comes close to the value $T_c = 154 \pm 8$ MeV suggested by a recent lattice simulation [7]. If we include only the transversal gluon degrees of freedom for $\chi=0$ (see Sec. IV) the value of T_c becomes somewhat larger by about 4%.

VI. HIGH-TEMPERATURE PHASE

The results of a numerical evaluation of $U(\chi, T)$ can be understood qualitatively by simple analytical considerations. For large temperature $T > M$ the integrals J are dominated by their values at $M=0$:

$$J_B(0, T) = J_G(0, T) = -\frac{\pi^2}{90} T^4, \quad J_F(0, T) = \frac{7\pi^2}{720} T^4. \quad (23)$$

If the minimum of U for large T occurs at $\chi=0$, the gauge bosons and quarks are effectively massless. The pressure is then given by

$$p = -U(\chi=0) = \frac{16\pi^2}{90} T^4 + \frac{12N_f\pi^2}{90} \left(\frac{7}{8}\right) T^4 - U_0(0). \quad (24)$$

We recognize the contribution of a free gas of $2(N_c^2 - 1) = 16$ bosonic and $4N_c N_f = 12N_f$ fermionic massless degrees of freedom. The scalar part $\Delta_s U$ is subleading and will be neglected (see Sec. IV). The negative contribution $-U_0(0) = -\lambda\chi_0^4$ accounts in our approximation for interaction effects related to the octet condensation. It becomes irrelevant only for large T whereas for temperatures near T_c its effects cannot be neglected. For large T the pressure approaches rapidly the Stefan-Boltzmann limit

$$p_{SB} = \left(\frac{8}{45} + \frac{7}{60} N_f\right) \pi^2 T^4. \quad (25)$$

The energy density

$$\epsilon = U - T \frac{\partial U}{\partial T} = U_0(0) + \frac{8}{15} \left(1 + \frac{21N_f}{32}\right) \pi^2 T^4 \quad (26)$$

deviates from the one for a relativistic gluon-quark gas due to U_0 . One obtains the equation of state

$$\epsilon - 3p = 4U_0(0), \quad \frac{\epsilon - 3p}{\epsilon + p} = \frac{45}{\left(8 + \frac{21}{4} N_f\right) \pi^2} \frac{U_0(0)}{T^4}. \quad (27)$$

This yields, for $N_f = 3$,

$$\tau = \frac{\epsilon - 3p}{\epsilon + p} \approx 0.192 \frac{U_0(0)}{T^4} = \tau(T_c) \frac{T_c^4}{T^4}, \quad \frac{p}{\epsilon} = \frac{1 - \tau}{3 + \tau}. \quad (28)$$

We have indicated the value of $\tau(T_c)$ obtained from the numerical evaluation of Eq. (16) in the chiral limit in Table I (and for realistic quark masses in Table II in Sec. XI). The small value of τ for the instanton-induced potential (A) implies that the equation of state for a relativistic plasma is particularly rapidly approached as the temperature increases beyond T_c . The fast turnover to a relativistic plasma is consistent with findings in lattice gauge theories [2].

In a more accurate treatment the free energy of the quark gluon gas and the Stefan-Boltzmann law receive additional perturbative corrections also for large T . The dominant corrections can be incorporated in our picture by adding the temperature dependent mass terms in Eq. (11) [21]. We emphasize, however, that the contribution $-U_0(0)$ in Eq. (24) remains as an important nonperturbative effect.

VII. CHIRAL PHASE TRANSITION

For a first-order transition the critical temperature T_c can be related to $U_0(0)$. Let us denote the value of the potential in the hadronic phase at T_c by $U_{SSB} = U[\chi_0(T_c)]$. Equating this with Eq. (24) yields

$$T_c^4 = \frac{45}{8\pi^2} \left(1 + \frac{21N_f}{32}\right)^{-1} [U_0(0) - U_{SSB}], \quad (29)$$

where we have neglected $\Delta_s U$ since the Goldstone boson fluctuations are not relevant at $\chi=0$ for a sufficiently strong first-order transition. As long as T_c remains much smaller than the mass of the gauge bosons $\bar{\mu}_\rho^{(SSB)}$ and fermions $M_q^{(SSB)}$ in the phase with broken chiral symmetry, their contribution to U_{SSB} remains strongly Boltzmann suppressed (see Table I). For a strong first-order phase transition the quantity U_{SSB} is therefore dominated by the fluctuations of the $N_f^2 - 1$ Goldstone bosons. If $\chi_0(T)$ is close to $\chi_0 = \chi_0(0)$ we can also neglect the difference $U_0[\chi_0(T)] - U_0(\chi_0)$ and U_{SSB} is given by the contribution of a free gas of Goldstone bosons

$$U_{SSB} = -\frac{(N_f^2 - 1)\pi^2}{90} T_c^4. \quad (30)$$

As a consequence, the critical temperature is given by $U_0(0)$ quite independently of the details of the shape of $U_0(\chi)$ or the thermal fluctuations

$$T_c^4 = \frac{45}{8\pi^2} \left(1 + \frac{21N_f}{32} - \frac{N_f^2 - 1}{16}\right)^{-1} U_0(0) = 0.231 \quad U_0(0). \quad (31)$$

(For the quantitative estimate we use $N_f = 3$.) For the approximation (28) this yields an equation of state

$$\frac{\epsilon - 3p}{\epsilon + p} \approx \tau(T_c) \frac{T_c^4}{T^4} \approx 0.83 \frac{T_c^4}{T^4} \quad (32)$$

and we see again that a free relativistic gas of gluons and quarks is reached rapidly as T increases beyond T_c . The pressure is continuous at T_c and directly given by $p(T_c) = -U_{SSB}$. Comparing Eq. (30) with the Stefan-Boltzmann value (25) yields $p(T_c)/p_{SB} = \frac{16}{95} (\frac{3}{37})$ for $N_f = 3(2)$.

Equation (31) relates the critical temperature to the properties of the η' meson and the fractional contribution R_{an} of the chiral anomaly to $U_0(0)$ [cf. Eq. (8)]

$$T_c = R_{an}^{-1/4} N_f^{-1/2} \times 221 \text{ MeV}. \quad (33)$$

For $R_{an} = (0.3, 0.5, 0.8, 1)$ and $N_f = 3$ this yields $T_c = (173, 152, 135, 128)$ MeV. Fluctuations in the SSB phase beyond the pions (30) further lower the value of U_{SSB} . This replaces in Eq. (33) R_{an} by \bar{R} , with $\bar{R} < R_{an}$, leading to an increase of T_c . For three-flavor QCD our numerical results are consistent with this qualitative picture of a strong first-order phase transition.

The quantitative estimates (31)–(33) are only valid if the minimum at $\chi_0(T_c)$ is still near χ_0 . This depends on the influence of the Goldstone bosons and, in particular, on the ratio between T_c and the meson decay constant f (see Sec. XI). For a weak first-order transition or a second-order transition (for $N_f = 2$) the potential difference $U_0[\chi_0(T_c)]$ has to be included in U_{SSB} (30). This contribution lowers the critical temperature. On the other hand, the gauge boson and quark fluctuations may not be negligible anymore for low enough $\chi_0(T_c)$. This effect enhances the critical temperature. The difference between the rough estimates (33) and (32) and the numerical computation (Table I) is related to these effects.⁶ It is particularly pronounced for the potential (A) for which the gauge boson mass in the SSB phase is already considerably smaller at T_c than at $T = 0$.

In the two-flavor case the chiral phase transition in QCD resembles in many respects the electroweak phase transition. The vacuum potential is analytic in χ^2 at $\chi^2 = 0$ such that a first-order transition can only be induced by the fluctuations of the gauge bosons. In this analogy the octet replaces the Higgs scalar and M_σ plays the role of the mass of the Higgs scalar M_H . Also the gluons replace the weak gauge bosons and the quarks play the role of the top quark in the electroweak theory. The electroweak phase diagram shows a first-order phase transition for small M_H and one may expect the same for QCD for small M_σ . In the electroweak theory this transition ends for some critical $M_{H,c}$ in a second-order transition, whereas for $M_H > M_{H,c}$ the transition is replaced by a continuous crossover [5,22,23]. In the case of QCD with vanishing quark masses one has, however, an additional important ingredient, namely, the existence of an order parameter and Goldstone bosons. In fact, in the

absence of current quark masses an order parameter $\chi_0(T) \neq 0$ breaks spontaneously a global symmetry. In consequence, there must be a phase transition as $\chi_0(T)$ reaches zero when the critical temperature T_c is reached from below. This holds for arbitrarily large M_σ . In two-flavor QCD with vanishing quark masses a second-order phase transition replaces the crossover of the electroweak theory. For nonzero quark mass the analogy is even closer, with a change from a first-order transition to crossover in dependence on M_σ . In contrast to the electroweak theory, however, M_σ is not a free parameter but can, in principle, be computed in QCD. An important ingredient for the determination of the characteristics of the phase transition is the size of the gauge coupling which is large in QCD. The realization of a second-order transition/crossover seems therefore quite plausible in two-flavor QCD.

VIII. SPONTANEOUS CHIRAL SYMMETRY BREAKING FOR $T < T_c$

For an understanding of the temperature effects in the hadronic phase for $T < T_c$ we need the value of $\chi_0(T)$. The octet expectation value in thermal equilibrium obeys the field equation

$$\frac{\partial U}{\partial \chi} = 2\chi \frac{\partial U}{\partial (\chi^2)} = 2\chi U' = j_\chi, \quad (34)$$

where $j_\chi = 0$ in absence of quark masses. For a determination of the temperature dependence of the order parameter $\chi_0(T)$ it is useful to investigate the field equation analytically. For the gluon contribution one finds from Eq. (17)

$$\Delta_g U' \equiv \frac{\partial}{\partial (\chi^2)} \Delta_g U = 12 \hat{g}^2 [\mu_\rho(T)] I_1^{(B)} [\mu_\rho^2(T), T]. \quad (35)$$

Here

$$\begin{aligned} \hat{g}^2 &= \frac{\partial \mu_\rho^2(T)}{\partial (\chi^2)} = g^2(\mu_T) \left\{ 1 - \frac{\mu_\rho^2}{\mu_\rho^2 + (\pi T)^2} \frac{(\beta_g/g)(\mu_T)}{(\beta_g/g)(\mu_\rho)} \right. \\ &\quad \left. \times \left(1 - \frac{g}{\beta_g}(\mu_\rho) \right)^{-1} \right\} \end{aligned} \quad (36)$$

reflects the running of the gauge coupling in dependence on the temperature-dependent gluon mass

$$\mu_\rho^2(T) = g^2(\mu_T) \chi^2. \quad (37)$$

The integral

⁶The numerical computation also uses 24 gauge boson degrees of freedom [Eq. (17)] instead of 16 for the analytical discussion in Secs. VII and VIII. This lowers T_c by 4%.

$$\begin{aligned}
I_1^{(B)}(M^2, T) &= 2 \frac{\partial}{\partial M^2} J_B(M^2, T) \\
&= T \sum_n \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{\vec{q}^2 + (2\pi n T)^2 + M^2} \\
&\quad - \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + M^2} \\
&= \frac{T^2}{2\pi^2} \int_0^\infty dy \frac{\sqrt{y^2 + 2my}}{e^{y+m} - 1}
\end{aligned} \tag{38}$$

contains for large T terms quadratic, linear and logarithmic in T [24]:

$$I_1^{(B)} \approx \frac{T^2}{12} - \frac{TM}{4\pi} + \frac{M^2}{16\pi^2} \ln \frac{(\tilde{c}T)^2 + M^2}{M^2}. \tag{39}$$

More precisely, the expression (39) applies for $M^2 \ll \pi^2 T^2$ and we will drop the logarithm in the following. We observe an infrared divergence in $\partial I_1^{(B)}/\partial M^2$ for $M \rightarrow 0$.

The fermionic contribution reads

$$\frac{\partial}{\partial(\chi^2)} \Delta_q U = -6N_f \hat{h}^2 I_1^{(F)}(h_\chi^2 \chi^2, T) \tag{40}$$

with

$$\begin{aligned}
\hat{h}^2 &= \frac{\partial}{\partial(\chi^2)} [h_\chi^2(\mu_T) \chi^2] \\
&= h_\chi^2(\mu_T) \left\{ 1 - \frac{\mu_\rho^2}{\mu_\rho^2 + (\pi T)^2} \frac{(\beta_h/h_\chi)(\mu_T)}{(\beta_g/g)(\mu_\rho)} \right. \\
&\quad \left. \times \left(1 - \frac{g}{\beta_g}(\mu_\rho) \right)^{-1} \right\}.
\end{aligned} \tag{41}$$

The corresponding integral

$$\begin{aligned}
I_1^{(F)}(M^2, T) &= 2 \frac{\partial}{\partial M^2} J_F(M^2, T) = - \frac{T^2}{2\pi^2} \int_0^\infty dy \frac{\sqrt{y^2 + 2my}}{e^{y+m} + 1} \\
&= \frac{1}{24} T^2 + \dots
\end{aligned} \tag{42}$$

involves half-integer Matsubara frequencies and does not show an infrared divergence.

The Goldstone-boson integral $I_1^{(G)} = 2 \partial J_G / \partial M^2$ equals I_1^B for $M^2 \geq 0$. For negative M^2 it becomes

$$\begin{aligned}
I_1^{(G)} &= - \frac{T^2}{2\pi^2} \int_0^\infty dy y (y^2 + |m^2|)^{-1/2} \ln(1 - e^{-y}) \\
&= \frac{T^2}{12} + \Delta I_1^{(G)}.
\end{aligned} \tag{43}$$

For small values of $|M|/T$ one finds

$$\Delta I_1^{(G)} = \frac{T|M|}{2\pi^2} \left(\ln \frac{2|M|}{T} - 1 \right) + \dots \tag{44}$$

and we see that $I_1^{(G)}$ is continuous at $M^2 = 0$. On the other hand, for large $|M|/T$ the result is

$$I_1^{(G)} = \frac{\hat{c}}{4\pi^2} \frac{T^3}{|M|} + \dots, \quad \hat{c} = \int_0^\infty dy \frac{y^2}{e^y - 1} \approx 2.4041. \tag{45}$$

For sufficiently large T a positive temperature-dependent masslike term dominates the potential derivative at the origin

$$\frac{\partial \Delta U}{\partial(\chi^2)}|_{\chi=0} = \left(\hat{g}^2 + \frac{N_f}{4} \hat{h}_\chi^2 \right) T^2. \tag{46}$$

This overwhelms any classical contribution and there is therefore no nontrivial minimum of $U(\chi, T)$ for $\chi \neq 0$. Chiral symmetry is restored [$\chi_0(T) = 0$] and the gluons no longer acquire a mass from the octet condensate. On the other hand, for $T < T_c$ the absolute minimum of the potential occurs for $\chi_0(T) \neq 0$, corresponding to a nontrivial solution of Eq. (34).

IX. GOLDSTONE-BOSON FLUCTUATIONS

For a determination of $\chi_0(T)$ in the low temperature phase with chiral symmetry breaking the Goldstone boson fluctuations are important. They dominate the low-temperature behavior and play an important role at a possible second-order phase transition. The correct treatment of the Goldstone bosons in a mean-field-type approach is rather subtle due to complications in the infrared physics for massless bosons. (This also applies to gauge bosons.) We show in the next section that our renormalization-group motivated prescriptions lead to the correct behavior at low T and we demonstrate in Appendix B that they also can account⁷ for the possibility of a second-order phase transition. The latter is particularly relevant for the case of two light quark flavors or for realistic QCD where the physical strange quark mass seems to correspond to a very weak first-order transition or a crossover, in the vicinity of a second-order transition for an nearby critical strange quark mass.

The Goldstone boson mass involves the derivative of the temperature-dependent effective potential $M_G^2 = c_G U'$, which obeys for all T

$$U' = B(\chi) + \frac{N_f^2 - 1}{2} c_G U'' \Delta I_1^{(G)}(c_G U', T) + \dots, \tag{47}$$

⁷A naive mean field treatment typically produces a first-order transition when massless bosons are present. This is due to the nonanalytic “cubic term” in the expansion of J_B [Eq. (19)] in powers of M .

$$B(\chi) = U'_0 + \frac{N_f^2 - 1}{24} c_G T^2 U''_0 + 12 \hat{g}^2 I_1^{(B)}(g^2 \chi^2, T) - 6 N_f \hat{h}^2 I_1^{(F)}(h^2 \chi^2, T).$$

Here the ellipses stand for corrections arising from ΔJ_G (A3) which vanish for $U' = 0$ and are negligible in the vicinity of an extremum. From $\Delta I_1^{(G)}(0, T) = 0$ one infers a simple equation for the extrema of U away from the origin. The condition $U' = 0$ is realized for $B = 0$ or

$$U'_0 + \frac{N_f^2 - 1}{24} c_G U''_0 T^2 = 6 N_f \hat{h}^2 I_1^{(F)}[h^2(\mu_T) \chi_0^2, T] - 12 \hat{g}^2 I_1^{(B)}[g^2(\mu_T) \chi_0^2, T], \quad (48)$$

where we recall the definitions

$$U'_0 = \frac{1}{2\chi} \frac{\partial U_0}{\partial \chi}, \quad U''_0 = \frac{1}{4\chi^2} \left(\frac{\partial^2 U_0}{\partial \chi^2} - \frac{1}{\chi} \frac{\partial U_0}{\partial \chi} \right). \quad (49)$$

For low temperature the right-hand side of Eq. (48) is Boltzmann suppressed and can be neglected.

The differential equation (47) for $U'(\chi)$ can be turned into an algebraic “Schwinger-Dyson equation” by replacing U'' by U''_0 . This is reasonable for $U''_0 \geq 0$, whereas for negative U''_0 we omit the contribution $\sim U''_0$ in B and $\Delta I_1^{(G)}$ in Eq. (47) such that $U' = B$. These prescriptions⁸ determine U' for given values of χ and T . Not too far from the minimum one may further approximate (for $U''_0 \geq 0$)

$$U' = B - \frac{N_f^2 - 1}{8\pi} c_G^{3/2} U''_0 T \left(\sqrt{U' \Theta(U')} - \frac{2}{\pi} \sqrt{-U' \Theta(-U')} \right) \times [\ln(2|U'|/T) - 1]. \quad (50)$$

A reasonable approximate formula for U' used for our numerical computation is given by

$$U' = \frac{1}{4} \text{sign}(B) (\sqrt{E^2 + 4|B|} - E)^2, \quad E = \frac{N_f^2 - 1}{8\pi} c_G^{3/2} T U''_0 \Theta(U''_0). \quad (51)$$

This implies that for small B in the vicinity of the minimum where $B \sim \chi - \chi_0$, one has⁹ $U' \sim (\chi - \chi_0)^2$.

⁸In regions of the potential where the gauge boson and fermion fluctuations are important a better approximation would replace $U''_0 \rightarrow U''_0 + \Delta_g U'' + \Delta_q U''$, similar to Eq. (A2).

⁹This holds for $E > 0$. Note that for $U' \sim \chi - \chi_0$ the nonanalyticity in $\Delta I_1^{(G)}$ would destabilize the minimum.

X. LOW TEMPERATURE PION GAS AND CHIRAL PERTURBATION THEORY

The thermodynamics for low temperature should be described by a gas of interacting pions. Any analytical computation which pretends to describe the transition from a pion gas at low T to a quark gluon plasma at high T should reproduce the low-temperature limit correctly. Indeed, in our formulation only the massless or very light particles contribute for low T . In particular, around the minimum of U near χ_0 both the baryons and the vector mesons (or, equivalently, the quarks and the gluons) are heavy and exponentially Boltzmann suppressed. Only the pions are light and one expects that they completely dominate the temperature effects at low T for χ near χ_0 .

The effects of a thermal interacting pion gas are described by chiral perturbation theory [25] which predicts for the chiral condensate¹⁰

$$\frac{\chi_0(T)}{\chi_0(0)} = \frac{\langle \bar{\psi} \psi \rangle(T)}{\langle \bar{\psi} \psi \rangle(0)} = 1 - \frac{N_f^2 - 1}{N_f} \frac{T^2}{12f^2} - \frac{N_f^2 - 1}{2N_f^2} \left(\frac{T^2}{12f^2} \right)^2 + \dots \quad (52)$$

Within the nonlinear setting of chiral perturbation theory one can understand the temperature effects of the interacting pions on the chiral condensate in a straightforward way. In particular, it is obvious that the lowest two orders can only depend on the ratio T/f [25]. In order to describe the phase transition and the high-temperature phase we are bound, however, to use a linear description for the scalar fields. The correct reproduction of lowest order chiral perturbation theory is not trivial in a linear setting and can be used as a test for the correctness of the particular formulation of the mean field approximation.

In our picture we can parametrize the effective potential in the vicinity of the minimum χ_0 by ($U = U_0 + \Delta U$)

$$U_0 = \frac{4}{3} M_\sigma^2 (\chi - \chi_0)^2, \quad \Delta U = (N_f^2 - 1) \left(-\frac{\pi^2}{90} T^4 + \frac{c_G}{24} T^2 \partial U_0 / \partial (\chi^2) \right). \quad (53)$$

Up to the χ -independent contribution $\sim T^4$ this yields

$$U = \frac{4}{3} M_\sigma^2 \left\{ (\chi - \chi_0)^2 + \frac{c_G (N_f^2 - 1)}{24} \frac{T^2}{\chi} (\chi - \chi_0) \right\} \quad (54)$$

and one infers that the T -dependent minimum is independent of M_σ

¹⁰Here we have neglected the T dependence of the wave function renormalization which relates χ and $\langle \bar{\psi} \psi \rangle$.

$$\frac{\chi_0(T)}{\chi_0(0)} = 1 - \frac{c_{GC_F}}{4}(N_f^2 - 1) \frac{T^2}{12f^2} \frac{\chi_0^2(0)}{\chi_0^2(T)}. \quad (55)$$

With $c_{GC_F} = 4/N_f$ [see Eq. (15)] we can verify that the lowest order of an expansion in $[\chi_0(T) - \chi_0(0)]/\chi_0(0)$ corresponds indeed to chiral perturbation theory. This shows the consistency of our treatment of the scalar fluctuations as discussed in Appendix A. A lack of care in the treatment of the scalar fluctuations easily leads to inconsistency with chiral perturbation theory even in lowest order. In fact, it is crucial that the term $\sim T^2$ in ΔU [Eq. (53)] involves the derivative of U_0 without additional temperature corrections to the potential.

XI. NONZERO QUARK MASSES

We finally discuss the effect of the quark masses in the limit where they are all equal. A quark mass term adds to the potential a linear¹¹ piece

$$U_j = U_0 + U_m, \quad U_m = -j_\chi(\chi - \hat{\chi}_0). \quad (56)$$

Therefore the location $\bar{\chi}_0$ of the minimum of $U + U_m$ obeys for arbitrary T

$$\frac{\partial U}{\partial \chi}(\bar{\chi}_0) = j_\chi. \quad (57)$$

Here j_χ is proportional to the current quark mass and can be related to the mass of the octet of light pseudo-Goldstone bosons (π, K, η) by

$$m_{PS}^2(T) = c_G \frac{\partial U}{\partial (\chi^2)}[\bar{\chi}_0(T)] = \frac{c_G}{2\bar{\chi}_0(T)} j_\chi. \quad (58)$$

In particular, $\bar{\chi}_0 = \bar{\chi}_0(0)$ is determined by the vacuum mass $m_{PS} = m_{PS}(0)$

$$U'_0(\bar{\chi}_0) = m_{PS}^2/c_G. \quad (59)$$

Vanishing pressure in the vacuum (at $T=0$) requires $U_0(\bar{\chi}_0) + U_m(\bar{\chi}_0) = 0$ or

$$\hat{\chi}_0 = \bar{\chi}_0 - U_0(\bar{\chi}_0)/j_\chi. \quad (60)$$

For a small quark mass (small j_χ) both $\bar{\chi}_0$ and $\hat{\chi}_0$ are given approximately by χ_0 .

Near the chiral limit we can understand various quark mass effects on the chiral phase transition analytically.

¹¹We omit nonlinear quark mass corrections from the chiral anomaly here. We also observe that the linear term actually appears for the $\bar{q}q$ -color singlet σ and is transmuted to χ only by integrating out σ . This results again in a nonlinearity of U_m . We consider U_m as an approximation for $T \leq T_c$. There is actually only a singlet and no octet condensate in the high temperature phase for nonvanishing quark masses.

(1) The vacuum expectation value χ_0 increases and $U_0(0) + U_m(0)$ acquires an additional positive contribution $j_\chi \hat{\chi}_0$. Both effects enhance T_c .

(2) In the high-temperature phase the minimum occurs at $\chi_s(T) \neq 0$. For a first-order transition this reduces the difference between U_j in the low- and high-temperature phases and therefore diminishes somewhat the increase of T_c . Also the quarks get a nonzero mass in the high-temperature phase $h_\chi \chi_s(T)$. This decreases the pressure of the fluctuations at given T and therefore enhances the value of T_c needed in order to compensate the difference in U_j . For small j_χ this effect is only quadratic in j_χ and the linear effect in $U_j[\chi_s(T)] - U_j[\chi_0(T)]$ dominates.

(3) As T_c and $\chi_s(T_c)$ increase with increasing j_χ , the difference between the two local minima characteristic for the first-order transition becomes less and less pronounced. Finally, the transition line ends for a critical $j_{\chi,c}$ at a second-order transition, with crossover for $j_\chi > j_{\chi,c}$. The situation is similar to the liquid-gas transition.

(4) In the case of a second-order transition for $j_\chi = 0$ the chiral transition turns to a crossover for all nonzero j_χ .

In summary, nonvanishing quark masses enhance the critical temperature and make transitions less pronounced. The case of realistic QCD with $m_s \neq m_{u,d}$ is somewhat more complicated since the expectation values in the high-temperature phase differ in the strange and nonstrange directions. In case of a strong first-order transition this modification is, however, only of little quantitative relevance. On the other hand, one may envisage a situation where for $m_s \neq 0, m_u = m_d = 0$ the critical behavior resembles two-flavor QCD.

Our numerical evaluation of $U(\chi, T)$ is easily exploited for nonvanishing quark masses. They only enter into the determination of the location of the minima at $\chi_s(T)$ and $\bar{\chi}_0(T)$ via Eq. (58). We indicate the vacuum properties for anomaly dominated potentials ($R_{an} = 0.9$) in Table II for the same values of the input parameters as used in Table I. Comparison between the two tables provides an estimate of the effect of nonzero degenerate quark masses. We employ [27] here a “realistic” average mass $m_{PS}^2 = (390 \text{ MeV})^2 \approx \frac{1}{3}(2M_K^2 + M_\pi^2)$ which corresponds to the neglect of $SU(3)$ -violating mass splittings in the pseudoscalar octet. We observe a sizeable increase of $f = \sqrt{7/9} \bar{\chi}_0$ and a corresponding increase in the baryon mass¹² $M_q = h_\chi(\bar{\mu}_\rho) \bar{\chi}_0$ and vector meson mass $\bar{\mu}_\rho = g(\bar{\mu}_\rho) \bar{\chi}_0$ in the vacuum.

For all three potentials (A),(B),(C) we find a first-order transition for $m_{PS} = 390 \text{ MeV}$. The critical temperature T_c is increased substantially as compared to the chiral limit, as can be inferred from Table II.

The mass effect on the critical temperature is moderate for our instanton-induced potential (A) for which the increase of

¹²Our choice of the Yukawa coupling leads to too large values of the baryon mass in vacuum for the potentials (B),(C). This is only of little quantitative relevance for the results presented in this note and can be corrected easily by the choice of a smaller Yukawa coupling.

about 10% is roughly consistent with a recent lattice simulation [7]. In this respect the instanton-induced potential computed in Ref. [14] does apparently much better than the two polynomial potentials (B),(C) that we use for comparison. We point out, however, that the neglect of the quark mass effects for the instanton-induced interaction leads to a substantial uncertainty in the quark mass dependence at the present stage.

XII. Z_3 VORTICES

In an attempt to find distinctive features of our scenario which could be tested by lattice simulations, we may look at topologically stable excitations. The octet condensate in the low-temperature phase indeed implies the stability of macroscopic Z_3 vortices. They are absent in the high-temperature phase. In the early universe, such strings would have been produced as topological defects during the QCD-phase transition. The string tension is typically of the order $\sigma \sim m_\rho^2$ such that $G\sigma \approx (m_\rho/M_\rho)^2$ is tiny. Observational consequences of the production of typically one string per horizon at the time of the phase transition are not obvious—in particular such strings are not candidates for seeds of a later galaxy formation.

As topological defects, the Z_3 vortices correspond to the nontrivial homotopy group $\pi_1[SU(3)/Z_3] = Z_3$. The color octet (and singlet) scalars transform trivially under the center of the color group, similar to the gluons. Any classical bosonic field configuration is therefore invariant under Z_3 transformations. Once the color group gets “spontaneously broken” by the octet condensate, the Z_3 vortices become topologically stable. As an example, consider a static vortex in the z direction with octet scalar field

$$\chi_{ijab} = \frac{1}{2\sqrt{6}} \chi(r) (\lambda_z)_{ab} [v^\dagger(\varphi) \lambda_z v(\varphi)]_{ji}. \quad (61)$$

Here v is given by a φ -dependent gauge transformation

$$v(\varphi) = \exp\left(\frac{i}{\sqrt{3}} \varphi \lambda_8\right) \quad (62)$$

such that a rotation around 2π corresponds to an element of Z_3

$$\exp\left(\frac{2\pi i}{\sqrt{3}} \lambda_8\right) = \exp\left(\frac{2\pi i}{3}\right). \quad (63)$$

The scalar field is therefore well defined and free of singularities provided $\chi(r)$ behaves properly at the origin, e.g., $\chi(r=0)=0$. For $r \rightarrow \infty$ we assume that $\chi(r)$ approaches the expectation value χ_0 which minimizes the effective potential, $\chi(r \rightarrow \infty) \rightarrow \chi_0$. The potential energy of the string is therefore concentrated in the core of the string, typically of radial size m_ρ . The nontrivial homotopy group tells us that no smooth deformation can change the nontrivial behavior for $r \rightarrow \infty$. In particular, no (singularity free) gauge transformation can “unwind” the string.

Without the gauge fields, the gradient energy per length of the configuration (61), namely, $E_{\text{grad}} \sim \int dr r^{-1} \partial_\varphi \chi_{ijab}^* \partial_\varphi \chi_{ijab} \sim \int dr r^{-1} \chi^2(r)$, would still diverge logarithmically for $r \rightarrow \infty$. We may supplement a gauge field in the φ direction which becomes a pure gauge for $r \rightarrow \infty$ [$a(r \rightarrow \infty) = 1, a(r=0) = 0$]

$$A_\varphi = -\frac{i}{g} \partial_\varphi v^T v^* a(r) = \frac{1}{\sqrt{3}g} \lambda_8 a(r). \quad (64)$$

It is then easy to see that the covariant derivative vanishes for $r \rightarrow \infty$:

$$D_\varphi \chi_{ijab} = \partial_\varphi \chi_{ijab} - ig(A_{ik})_\varphi \chi_{kjab} + ig \chi_{ikab} (A_{kj})_\varphi \rightarrow 0. \quad (65)$$

The gauge field strength vanishes as well in this limit. We are therefore sure that the vortex has a finite string tension, the precise value depending on the shape of the functions $\chi(r)$ and $a(r)$ which have to be determined by solving the field equations with the appropriate boundary values for $r=0$ and $r \rightarrow \infty$.

Actually, the topological situation which leads to stable vortices gets somewhat more involved by the fact that both color and flavor groups are broken simultaneously by the octet condensate. A color rotation in the Z_3 element of $SU(3)_c$ can be “unwound” by a φ -dependent color-flavor-locked rotation. This shifts the nontrivial topology from the color sector to the flavor sector, more precisely to the nontrivial center of the diagonal flavor transformations from $SU(3)_L \times SU(3)_R$. The topological stability due to the nontrivial homotopy group π_1 is not affected. Since QCD is invariant only under global flavor transformations the vortex is, however, not invariant under coordinate-dependent color-flavor-locked transformations. The vanishing of the gauge covariant kinetic term for $r \rightarrow \infty$ occurs only if the φ -dependent rotation of the vacuum state is associated to the color sector.

In this context it is useful to recall the global structure of the symmetry group of QCD with three massless quarks. For infinitesimal transformations the symmetry group is

$$G = SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B. \quad (66)$$

Global rotations in the center of $SU(3)_c$ and the center of the diagonal vectorlike flavor group $SU(3)_V$ correspond to identical phase rotations of the quark fields. These transformations belong also to the Abelian group $U(1)_B$ which is associated to conserved baryon number. A direct consequence of this global group structure is the triality rule: Color representations in the class $(3, \bar{6}, \text{etc.})$ have baryon number $B = 1/3 \bmod 1$ and electric charge $Q = 2/3 \bmod 1$ whereas the class $(1, 8, \text{etc.})$ has $B = 1 \bmod 1$ and $Q = 1 \bmod 1$. This explains why the physical fermions are baryons with $B = 1$ and integer Q , since all physical states are color singlets. (Some care is needed for the correct interpretation of the Higgs picture [11].)

In a first look it may seem that the macroscopic Z_3 vortices are not present in the confinement picture of QCD, in

contradiction to our assumption of equivalence of the Higgs and confinement pictures. This is, however, not obvious. In gluodynamics (QCD without light quarks) the Z_N vortices are discussed as important configurations for the understanding of confinement [29]. Lattice simulations could establish the existence of macroscopic vortices in the vacuum of three flavor QCD and their absence in the high-temperature phase. Such a finding (or the contrary) may be interpreted as an important test for our picture.

XIII. CONCLUSION

In conclusion, we have presented here a rather simple picture of the high-temperature phase transition in QCD. Both confinement and chiral symmetry restoration are associated to the melting of a color-octet condensate at the critical temperature. The main phenomenological features of our picture of the phase transition are the following.

(1) For three light quarks with equal mass a first-order phase transition separates a low-temperature “hadronic phase” from a high-temperature “quark-gluon phase.” In the chiral limit of vanishing current quark masses the quarks and gluons are massless in the high-temperature phase where chiral symmetry is restored. Color symmetry becomes a symmetry of the spectrum of pseudoparticles above the critical temperature T_c . Below T_c both chiral symmetry and color symmetry are spontaneously broken. Chiral symmetry breaking generates a mass for the quarks which now appear as baryons. Eight massless Goldstone bosons signal the global symmetry breaking. Local color symmetry breaking gives a mass to the gluons by the Higgs mechanism. The gluons are identified with the octet of vector mesons (ρ, K^*, ω). The first-order transition is therefore associated with a jump in the vector meson mass (they become massless gluons for $T > T_c$) and the baryon mass (they turn to massless quarks for $T > T_c$). Our picture of gluon-meson duality and quark-baryon duality allows us to describe the excitations relevant above and below T_c by the same fields. The first-order phase transition extends to nonzero current quark mass, including values which lead in the vacuum to a “realistic” average mass for the (π, K, η) pseudoscalars $m_{PS} = 390$ MeV $\approx \sqrt{(2M_K^2 + M_\pi^2)}/3$.

(2) The effective multiquark interactions at low momentum or the corresponding effective potential for $q\bar{q}$ bilinears are presumably dominated by the axial anomaly arising from instanton effects. We can then relate the critical temperature of a strong first-order transition to the mass and decay constant of the η' meson. For $m_{PS} = 390$ MeV we find $T_c \approx 170$ MeV whereas the critical temperature is lower in the chiral limit. For an anomaly dominated vacuum potential the prediction for the chiral limit ($m_{PS} = 0$) is independent of many details and found in the range $T_c \approx 130 - 160$ MeV.

(3) A dynamical picture for an instanton-induced color octet condensate has been developed recently [14]. This has led to a computation of the octet potential which involves the running gauge coupling as the only free parameter. The characteristic features of the instanton-induced vacuum effective potential are depicted by the potential (A), Eq. (4). We concentrate in the following on this estimate. It is encouraging

that the predictions of the instanton-induced potential for the critical temperature, namely, $T_c = 170$ MeV for $m_{PS} = 390$ MeV and $T_c = 155$ MeV for $m_{PS} = 0$, agree well with recent lattice simulations [7]. One may interpret this as a test for the idea of spontaneous color symmetry breaking by instanton effects.

(4) For the equation of state in the quark-gluon phase we find at T_c a ratio between pressure and energy density $p/\epsilon \approx 0.13$ for $m_{PS} = 390$ MeV and $p/\epsilon \approx 0.19$ for $m_{PS} = 0$. This ratio approaches very rapidly the equation of state of a relativistic gas ($p/\epsilon = 1/3$) as T increases beyond T_c .

(5) In the hadronic phase the screening masses of baryons, vector mesons, and pseudoscalars show a strong temperature dependence as T approaches T_c from below. Near T_c the chiral symmetry breaking contribution to the average baryon mass $M_q \approx 600$ MeV is only about one half of the vacuum mass and approaches a typical constituent quark mass. Also the Higgs contribution to the ρ -meson mass is found substantially smaller than in the vacuum $\bar{\mu}_\rho(T_c) \approx 300$ MeV. For $T > T_c$ the effective chiral symmetry breaking fermion mass jumps to a small value $M_q \approx 40$ MeV which is compatible for $m_{PS} = 390$ MeV with common estimates for the average current quark mass $m_q = \frac{1}{3}(m_s + m_d + m_u)$. The average pseudoscalar mass is found as $m_{PS}(T_c) \approx 180$ MeV. It is therefore substantially smaller than in the vacuum.

The present paper also contains a rather detailed discussion of a mean-field-type computation of the temperature dependence of the effective potential. This has become necessary since a too naive treatment of the scalar fluctuations has previously often led to incorrect results near second-order or weak first-order phase transitions. The reason is the complicated infrared behavior of massless boson fluctuations in thermal equilibrium. A second problem is the nonconvexity of the perturbative effective potential which leads naively to negative scalar mass terms. Most previous work has simply left out the scalar fluctuations because of the technical difficulties associated with these problems. Nevertheless, the Goldstone boson fluctuations are important for QCD at low temperature and near the phase transition, and we therefore want to include them. Our prescriptions are based on lessons from earlier more involved renormalization-group investigations and should include the most dominant effects. In particular, they account correctly for the low temperature behavior as described by chiral perturbation theory and are consistent with the physics of a second-order phase transition if this occurs. Our treatment can easily be taken over for other models.

Our approach to the high-temperature phase transition in QCD still contains many uncertainties, which have been discussed at various places in this paper. In particular, a better understanding of the effective potential in the vacuum and a separate treatment of the color octet and singlet condensates would be most welcome and probably needed for a study of finer aspects of the phase transition. This concerns, in particular, the interesting question about the existence and order of a phase transition in “real QCD” with different strange quark and up/down quark masses. An appropriate tool for this purpose seems to be a renormalization group study simi-

lar to the successful treatment [20] of the phase transition in the Nambu–Jona-Lasinio model.

Nevertheless, we find the simple overall picture of the chiral and deconfinement transition rather convincing and the quantitative results of a first rough computation encouraging. We hope that this work can become a starting point for a quantitative analytical understanding of the QCD phase transition.

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APPENDIX A: RUNNING COUPLINGS

In general, the effective masses M appearing in the J integrals (18) depend on the momentum \vec{q}^2 . This could be expressed through a momentum dependence of the effective gauge and Yukawa couplings. We want to avoid here the complications of momentum-dependent masses. Instead, the dominant effects of the running couplings are taken into account by the choice of an appropriate renormalization scale μ in Eq. (11). Corrections to this approximation are reflected in the term ΔJ_G in Eq. (17).

For not too large m the integrals (19) are dominated by momenta $q^2 \approx (\pi T)^2$. We therefore may choose an appropriate temperature-dependent renormalization scale

$$\mu_T = \sqrt{\mu_\rho^2 + \pi^2 T^2}, \quad \mu_\rho^2 = g^2(\mu_\rho) \chi^2. \quad (\text{A1})$$

[Here $\mu_\rho^2 = \mu_\rho^2(\mu = \mu_\rho)$ refers to the vacuum with arbitrary χ corresponding to a possible presence of “sources” or quark mass terms.] This choice of μ_T is a valid approximation for the fermion fluctuations and we therefore use $h_\chi^2(\mu_T)$ in the argument of J_F in Eq. (17).

For bosons the issue is more involved due to the infrared behavior of classical statistics. The first two terms in a Taylor expansion $J_B(0)$ and $\partial J_B / \partial M^2|_{M^2=0}$ are dominated again by momenta $\vec{q}^2 \approx (\pi T)^2$. The remaining integral $\hat{J}_B = J_B - J_B(0) - M^2(\partial J_B / \partial M^2)|_{M^2=0}$ receives, however, an essential contribution from the $n=0$ Matsubara frequency which corresponds to classical statistics. The three-dimensional momentum integrals of classical statistics are more infrared singular than the four-dimensional integrals for the vacuum. In consequence, the remaining integral \hat{J}_B is dominated by momenta $\vec{q}^2 \approx M^2$. This leads in perturbation theory to the so-called cubic term [24] which has played an important role in the discussion of the electroweak phase transition [28]. Since J_B is not analytic at $M^2=0$, a careful treatment of the long-distance physics is mandatory if one aims for quantitative precision. For first-order phase transitions the momentum de-

pendence of M^2 has only moderate effects for the gauge boson fluctuations¹³ and we choose $M^2 = g^2(\mu_T) \chi^2$ in J_B .

For the Goldstone boson fluctuations more care is needed if one wants to be consistent with chiral perturbation theory for low T and the correct behavior at a second-order phase transition. We first include the gauge boson and fermion thermal fluctuations. At this level the mass term for the Goldstone bosons is given by

$$M_{G,0}^2 = c_G \frac{\partial}{\partial(\chi^2)} (U_0 + \Delta_g U + \Delta_q U). \quad (\text{A2})$$

In a second step we take the scalar fluctuations into account. Then only the complete mass term M_G^2 (12) vanishes at the minimum of $U(\chi, T)$. The thermally corrected (pseudo)particle mass term M_G^2 is relevant for momenta $\vec{q}^2 \lesssim |M_G^2|$ and we use M_G^2 for the argument of J_G . On the other hand, the high momentum part of the integral J_G is dominated by momenta where the thermal mass corrections from the scalar fluctuations are not yet important. Therefore $M_{G,0}^2$ (A2) is relevant in this momentum range. We account for this by the correction

$$\Delta J_G = \frac{T^2}{24} (M_{G,0}^2 - M_G^2) f_G(M_G^2/T^2), \quad (\text{A3})$$

which effectively replaces M_G^2 by $M_{G,0}^2$ for the term $\sim T^2$ which is dominated by momenta $\vec{q}^2 \approx (\pi T)^2$ if $T^2 > |M_G^2|$. On the other hand, the factor f_G reflects the fact that all temperature fluctuations and therefore also ΔJ_G are exponentially Boltzmann suppressed for $T^2 \ll M_G^2$. We choose here a simple form ($\gamma=0.5$) which mimicks Eq. (21) for large m^2

$$f_G(m^2) = \exp\{\gamma - [\gamma^4 + (m^2)^2]^{1/4}\} \left(1 + \frac{(m^2)^2}{\gamma^4}\right)^{-1/8}. \quad (\text{A4})$$

These prescriptions may seem somewhat *ad hoc* and technical. They find a deeper motivation from the study of the renormalization flow in thermal field theories [19,20]. We will see below that they guarantee agreement with chiral perturbation theory for low temperature and avoid unphysical singularities.

We finally have to specify the scale dependence of the gauge and Yukawa coupling. For the running gauge coupling we employ the perturbative β function in three-loop order in the $\overline{\text{MS}}$ scheme

$$\beta_g = \mu \frac{\partial}{\partial \mu} g = -\beta_0 \frac{g^3}{16\pi^2} - \dots, \quad \beta_0 = 11 - 2N_f/3. \quad (\text{A5})$$

¹³For a second-order phase transition one may split the momentum integral in J_B and use $M^2 = g_T^2(\mu_\rho) \chi^2$ for $\vec{q}^2 < (\mu_\rho/2)^2$. Here $g_T(\mu_\rho)$ obeys an effective three-dimensional evolution for $\mu_\rho < \pi T$ [5].

Our normalization $g(770 \text{ MeV})=6$ is suggested by the phenomenology of ρ decays into pions and $\mu^+ \mu^-$ [11]. For the Yukawa coupling we use the one-loop renormalization-group equation

$$\frac{dh_\chi^2}{d \ln \mu} = 2\beta_h h_\chi = \frac{a}{16\pi^2} h_\chi^4 - \frac{b}{16\pi^2} g^2 h_\chi^2. \quad (\text{A6})$$

Treating also the gauge coupling in one-loop order this has the solution

$$h_\chi(\mu) = h_\chi(\mu_0) \left(\frac{g^2(\mu)}{g^2(\mu_0)} \right)^{b/4\beta_0} \left\{ 1 + \frac{ah_\chi^2(\mu_0)}{(2\beta_0 - b)g^2(\mu_0)} \right. \\ \left. \times \left[1 - \left(\frac{g^2(\mu)}{g^2(\mu_0)} \right)^{b/2\beta_0 - 1} \right] \right\}^{-1/2}. \quad (\text{A7})$$

We use the approximation (A7) with $b=16$ such that we recover for small h_χ the standard anomalous dimension for the mass and take somewhat arbitrarily¹⁴ $a=1$. The normalization is fixed by Eq. (11), i.e., $h_\chi(770 \text{ MeV}) = 1.15 \text{ GeV}/\chi_0$.

APPENDIX B: SECOND-ORDER PHASE TRANSITION

For an investigation of a possible second-order phase transition in the chiral limit we concentrate on the behavior of U near $\chi^2=0$ or small values of χ^2/T^2 . In the vicinity of the critical temperature of a second-order (or weak first-order) transition the Goldstone fluctuations play a role in this region. The derivative of the temperature-dependent effective potential becomes (in the range where $U' \geq 0$)

$$U'(\chi, T) = U'_0 + \left(\hat{g}^2(\mu_T) + \frac{N_f \hat{h}_\chi^2(\mu_T)}{4} + \frac{N_f^2 - 1}{24} c_G U_0'' \right) T^2 \\ - \left(\frac{3g^3 \chi}{1 - \beta_g/g}(\mu_\rho) + \frac{N_f^2 - 1}{8} c_G^{3/2} U'' \sqrt{U'} \right) \frac{T}{\pi}. \quad (\text{B1})$$

Here we have improved our treatment of the gauge boson fluctuations by using $g(\mu_\rho)$ instead of $g(\mu_T)$ for the low momentum fluctuations corresponding to classical statistics (the term $\sim T$). Let us denote by T_0 the temperature where U' vanishes at $\chi=0$, i.e., $U'(0, T_0)=0$. A second-order phase transition at $T_c=T_0$ occurs if $U'(\chi, T_0)$ is strictly positive for all $\chi>0$. In contrast, one has a first-order transition if $U'(\chi, T_0)$ is negative for small $\chi>0$. Then the point $\chi=0$ corresponds to a maximum of $U(\chi, T_0)$ and the critical temperature for the first-order transition is below T_0 .

Consider first the potential (C) which is relevant for the two-flavor case. Omitting for a moment the Goldstone boson contributions and the running of the gauge coupling (β_g

$=0$), the term $\sim g^3 \chi T$ in Eq. (B1) would lead to a first-order transition. This corresponds to the well-known “cubic” term in the effective potential for the Higgs scalar in the electroweak theory. As has been discussed extensively in connection with the electroweak phase transition [22], the infrared physics of the gauge bosons responsible for this term has to be handled with care. In particular, the running of the gauge coupling is crucial for a correct picture. For a running g the product $g\chi = \mu_\rho$ goes to a constant $\Lambda(T)$ for $\chi \rightarrow 0$ and the nonanalyticity in χ^2 disappears. We note that χ -independent terms in Eq. (B1) only influence the location of T_0 . Denoting $U''(0, T_0) = g_\chi$ one obtains for small χ and $|T - T_0|$ an expansion, with constants c_1 , c_2 , and $N_f^2 - 1 = N$,

$$U' = g_\chi \chi^2 + c_1(T^2 - T_0^2) + c_2(T - T_0) - \frac{N c_G^{3/2}}{8\pi} g_\chi T \sqrt{U'}. \quad (\text{B2})$$

For $T \rightarrow T_0$ and $\chi \rightarrow 0$ this yields [26]

$$U'(\rho, T) = \left[\frac{8\pi}{N c_G^{3/2} T_0} [\chi^2 + c_3(T - T_0)] \right]^2 \quad (\text{B3})$$

and describes¹⁵ a second-order phase transition. The critical exponent $\nu=1$ corresponds¹⁶ to the leading order of the $1/N$ expansion in the effective three-dimensional theory [26]. An improvement of the description of the critical behavior (with more realistic critical exponents) needs to incorporate the fluctuations of the neglected scalar modes (see Sec. IV) and can be done with the help of modern renormalization group methods [19]. We conclude that a second-order chiral phase transition for $N_f=2$ is compatible with our picture, provided the infrared fluctuations of the gauge bosons are treated in the appropriate way.

For three massless flavors the presence of an effective cubic term $\sim \chi^3$ in the effective potential at zero temperature [Eq. (5)] changes the situation profoundly. This term reflects the axial anomaly and is induced by instanton effects. The term $\sim -\chi$ always dominates the right-hand side of Eq. (B1) for small enough values of χ since there is no competing term linear in χ . One infers the existence of a first-order phase transition for three massless flavors.

The vicinity of the critical temperature of a second-order or weak first-order transition is governed by universal behavior. The universality class is characterized by the symmetry and the representations which remain (almost) massless at T_c . For two-flavor QCD it is not obvious which are the massless scalar representations at a second-order phase transition. Furthermore, the gauge bosons behave essentially as a free gas near the transition. Even though their low momentum classical fluctuations may acquire an effective mass, they will influence the nonuniversal critical behavior. This

¹⁴A more accurate treatment would keep the color octet and singlet as separate fields. The evolution equations for the two respective Yukawa couplings differ.

¹⁵A first-order transition is possible if a new minimum appears before the running g has reached the behavior $g\chi = \text{const}$.

¹⁶This differs from high-temperature perturbation theory or chiral perturbation theory which would yield a “mean field exponent” $\nu = 1/2$.

influence of the gauge bosons is characteristic for a simultaneous chiral and deconfinement transition and leads to modifications of results obtained in quark-meson or Nambu–Jona-Lasinio models [20]. In view of the distance scales probed in lattice simulations, we find it rather unlikely that critical be-

havior in the standard $O(4)$ -universality class will show up there. This is independent of the interesting question what would be the true universality class of a second-order transition in two-flavor QCD, as seen at very large correlation length close to T_c .

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- [1] L. McLerran and B. Svetitsky, Phys. Lett. **98B**, 195 (1981); J. Kuti, J. Polonyi, and K. Szlachanyi, *ibid.* **98B**, 199 (1981); J. Engels, F. Karsch, I. Montvay, and H. Satz, *ibid.* **101B**, 89 (1981).
 - [2] F. Karsch, in *Proceedings of Strong and Electroweak Matter '98*: Copenhagen, Denmark, edited by J. Ambjorn *et al.* (World Scientific, Singapore, 1999), p. 101, hep-lat/9903031.
 - [3] See H. Meyer Ortmanns, Rev. Mod. Phys. **68**, 473 (1996), and references therein.
 - [4] T. Appelquist and R. Pisarski, Phys. Rev. D **23**, 2305 (1981); S. Nadkarni, *ibid.* **27**, 917 (1983); N.P. Landsman, Nucl. Phys. **B322**, 498 (1989); P. Lacock, D.E. Miller, and T. Reisz, *ibid.* **B369**, 501 (1992); L. Kärkkäinen, P. Lacock, B. Peterson, and T. Reisz, *ibid.* **B395**, 733 (1993).
 - [5] M. Reuter and C. Wetterich, Nucl. Phys. **B408**, 91 (1993).
 - [6] O. Philipsen, M. Teper, and H. Wittig, Nucl. Phys. **B528**, 379 (1998).
 - [7] F. Karsch, E. Laermann, and A. Peikert, Nucl. Phys. **B605**, 579 (2001).
 - [8] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); H. Hatsuda and T. Kunihiro, Phys. Rep. **247**, 221 (1994); J. Bijnens, *ibid.* **265**, 369 (1996).
 - [9] Proceedings of “Quark Matter ’97,” edited by T. Hatsuda *et al.* [Nucl. Phys. **A638** (1998)]; “Strangeness in Quark Matter ’98,” edited by M. Morando [J. Phys. G **25** (1998)]; “Quark Matter ’99,” edited by L. Riccati *et al.* [Nucl. Phys. **A661** (1999)].
 - [10] F.R. Brown *et al.*, Phys. Rev. Lett. **65**, 2491 (1990); A. Ali Khan *et al.*, Phys. Rev. D **63**, 034502 (2001).
 - [11] C. Wetterich, Phys. Lett. B **462**, 164 (1999); Phys. Rev. D **64**, 036003 (2001).
 - [12] J. Berges and C. Wetterich, Phys. Lett. B **512**, 85 (2001).
 - [13] M. Alford, K. Rajagopal, and F. Wilczek, Nucl. Phys. **B537**, 443 (1999); T. Schäfer and F. Wilczek, Phys. Rev. Lett. **82**, 3956 (1999); M. Alford, J. Berges, and K. Rajagopal, Nucl. Phys. **B558**, 219 (1999); T. Schäfer and F. Wilczek, Phys. Rev. D **60**, 074014 (1999).
 - [14] C. Wetterich, Phys. Lett. B **525**, 277 (2002).
 - [15] G.E. Brown and M. Rho, Phys. Rep. **363**, 85 (2002); M. Harada and K. Yamawaki, Phys. Rev. Lett. **87**, 152001 (2001).
 - [16] C. Wetterich, Eur. Phys. J. C **18**, 577 (2001).
 - [17] J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1989).
 - [18] A. Ringwald and C. Wetterich, Nucl. Phys. **B334**, 506 (1990); C. Wetterich, *ibid.* **B352**, 529 (1991); N. Tetradis and C. Wetterich, *ibid.* **B383**, 197 (1992); J. Berges, N. Tetradis, and C. Wetterich, Phys. Lett. B **393**, 387 (1997).
 - [19] C. Wetterich, Phys. Lett. B **301**, 90 (1993); N. Tetradis and C. Wetterich, Nucl. Phys. **B398**, 659 (1993); J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002).
 - [20] J. Berges, D. Jungnickel, and C. Wetterich, Phys. Rev. D **59**, 034010 (1999).
 - [21] A. Peshier, B. Kämpfer, O. Pavlenko, and G. Soff, Phys. Rev. D **54**, 2399 (1996); P. Levai and U. Heinz, Phys. Rev. C **57**, 1879 (1998).
 - [22] C. Wetterich, in *Electroweak Physics and the Early Universe*, edited by J. Romao and F. Freire (Plenum Press, New York, 1994), p. 229; B. Bergerhoff and C. Wetterich, in *Current Topics in Astrophysical Physics*, edited by N. Sanchez and A. Zichichi (World Scientific, Singapore, 1997), p. 132.
 - [23] W. Buchmüller and O. Philipsen, Nucl. Phys. **B443**, 47 (1995); K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, Phys. Rev. Lett. **77**, 2887 (1996).
 - [24] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
 - [25] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).
 - [26] M. Reuter, N. Tetradis, and C. Wetterich, Nucl. Phys. **B401**, 567 (1993).
 - [27] D. Jungnickel and C. Wetterich, Eur. Phys. J. C **1**, 669 (1998).
 - [28] A. Bochkarev and M. Shaposhnikov, Mod. Phys. Lett. A **2**, 417 (1987).
 - [29] L. Del Debbio, M. Faber, J. Greensite, and S. Olejnik, Phys. Rev. D **55**, 2298 (1997); M. Engelhardt, K. Langfeld, H. Reinhardt, and O. Tennert, *ibid.* **61**, 054504 (2000); C. Alexandrou, M. D’Elia, and P. de Forcrand, Nucl. Phys. **A663**, 1031 (2000); M. Faber, J. Greensite, and S. Olejnik, J. High Energy Phys. **06**, 041 (2000).